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An Improved Upper Bound in the Maximum Dispersal Problem

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ABSTRACT

For integral $m \geqslant 2$, let $x_1, \ldots, x_m$ be any unit vectors in R_n, the real Euclidean space of n dimensions. We obtain an upper bound for the quantity $\min_{i \neq j} \|x_i - x_j\|$ which, though not as simple, is uniformly sharper than one recently obtained by the author. The result has application to the so-called maximum-dispersal problem, an open problem recently popularized by Klee.

1. INTRODUCTION

For integral $m \geqslant 2$, put[1]

$$d_n(m) = \max_{\substack{x_1, \ldots, x_m \in R_n \\ \|x_i\| = 1 (1 \leqslant i \leqslant m)}} \min_{i \neq j} \|x_i - x_j\|. \tag{1}$$

We ask for the values of the $d_n(m)$, and for m unit vectors $x_1, \ldots, x_m \in R_n$ whose minimum inter-vector distance is the maximum value $d_n(m)$.

The problem described above, originally proposed in 1930 in the setting $n = 3$, has been solved, essentially, by Rankin [2] in 1954 for the cases $m \leqslant 2n$. Moore [1] has recently obtained some simple upper bounds for the $d_n(m)$ in the remaining cases; in particular, it has been shown in [1] that for any $n \geqslant 2$ and

[1] We use "$\|x\|$" to denote the usual Euclidean length of the vector x; the associated inner product of vectors x and y is denoted by "$(x|y)$."

$$m \geqslant \begin{cases} \sqrt{2\pi n} & \text{if } n \text{ is even} \\ \sqrt{2\pi\sqrt{e}\,(n-1)} & \text{if } n \text{ is odd} \end{cases} \tag{2}$$

we have

$$d_n(m) \leqslant 2\left[\frac{2\sqrt{\pi}}{m}\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right]^{1/(n-1)} = M_n(m). \tag{3}$$

Our purpose here is to show that the bound $M_n(m)$ in (3) may, with some loss in computational simplicity, be replaced by a uniformly sharper upper bound for the $d_n(m)$. The new bound is, moreover, valid for all $n \geqslant 2$ and $m \geqslant 2$.

Previous approaches to problems of this type have been hampered by the lack of simple formulas for the area of a spherical cap on the surface of the unit sphere S_n of R_n as a function of its angular diameter. We shall avoid this technical difficulty by developing a convenient procedure for computing the area of such caps recursively.

2. Results

By a spherical cap on S_n with center P and angular diameter θ, which we term a θ-cap, we mean the set of all points Q on the surface of S_n for which the angle QOP is less than or equal to $\theta/2$. We denote by $\omega_n(\theta)$ the area of a θ-cap on S_n.

Main Theorem. *For integral n and $m \geqslant 2$,*

$$d_n(m) \leqslant \left[2\left(1-\cos\left[\omega_n^{-1}\left(\frac{2}{m}\frac{\pi^{n/2}}{\Gamma(n/2)}\right)\right]\right)\right]^{1/2} = N_n(m). \tag{4}$$

Remarks. It is in general necessary to resort to numerical methods to evaluate the inverse function $\omega_n^{-1}(\cdot)$. A recursive formula for $\omega_n(\cdot)$, which the author has found useful for this purpose, is provided in Theorem 2.

The bound $N_n(m)$ in (4), though computationally less simple than $M_n(m)$, is uniformly sharper; that is $N_n(m) \leqslant M_n(m)$ for all m and n for which $M_n(m)$ is defined. A comparison of $N_n(m)$ and $M_n(m)$ for the case $n = 4$ (which is typical) is shown in Fig. 1; the figure also shows the exact values of the $d_4(m)$, so far as they are known. More complete tables of values of $N_n(m)$ have been compiled by the author and are available to readers upon request.

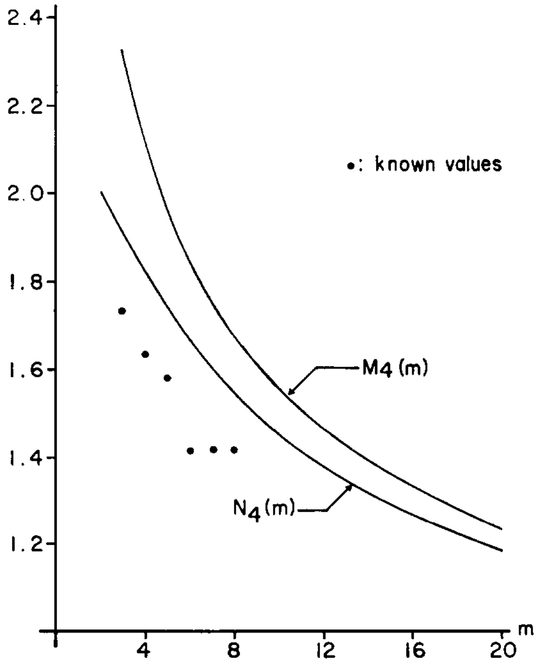

FIG. 1. Comparison of $M_4(m)$ and $N_4(m)$.

THEOREM 2. *For any* $0 \leqslant \theta \leqslant 2\pi$ *and* $n \geqslant 4$,

$$\omega_n(\theta) = \frac{2}{n-2}\left[\pi \cdot \omega_{n-2}(\theta) - \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \sin^{n-3}(\theta/2)\cos(\theta/2)\right]. \quad (5)$$

The result (5), together with the well-known formulas

$$\begin{aligned} \omega_2(\theta) &= \theta, \\ \omega_3(\theta) &= 2\pi \cdot [1 - \cos(\theta/2)], \end{aligned} \quad (6)$$

provides a quick and easy means of numerical computation of any $\omega_n(\theta)$. The following table lists the formulas which result from (5) and (6) for small n.

n	$\omega_n(\theta)$
2	θ
3	$2\pi[1 - \cos(\theta/2)]$
4	$\pi[\theta - \sin(\theta)]$
5	$\frac{4}{3}\pi^2[1 - \cos(\theta/2) - \frac{1}{2}\sin^2(\theta/2)\cos(\theta/2)]$
6	$\pi^2[\theta/2 - \sin(\theta/2)\cos(\theta/2) - \frac{2}{3}\sin^3(\theta/2)\cos(\theta/2)]$

3. PROOFS

We work first on Theorem 2. We require the well-known (see [3, p. 130]) formulas for the volume $V_n(r)$ and surface area $A_n(r)$ of an n-sphere $S_n(r)$ of radius r

$$V_n(r) = \frac{\pi^{n/2}}{\Gamma((n/2)+1)}r^n,$$

$$A_n(r) = 2\frac{\pi^{n/2}}{\Gamma(n/2)}r^{n-1}.$$

Consider a θ-cap on $S_n(1) = S_n$. For $n \geqslant 2$, the surface area formula of calculus gives

$$\omega_n(\theta) = \int_0^{\theta/2} \frac{1}{\cos\gamma}\, dV_{n-1}(\sin\gamma) = 2\frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\int_0^{\theta/2}\sin^{n-2}(\gamma)\, d\gamma. \tag{7}$$

Using the identity

$$\int_0^{\alpha}\sin^k(u)\, du = \frac{k-1}{k}\int_0^{\alpha}\sin^{k-2}(u)\, du - \frac{1}{k}\sin^{k-1}(\alpha)\cos(\alpha),$$

we may write (7) in the form

$$\omega_n(\theta) = 2\frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\left[\frac{n-3}{n-2}\int_0^{\theta/2}\sin^{n-4}(\gamma)\, d\gamma - \frac{1}{n-2}\sin^{n-3}(\theta/2)\cos(\theta/2)\right],$$

and now use of (7) once again gives the result (5).

The proof of the main theorem follows the pattern of that of (3) in [1]. For fixed $m \geqslant 2$, let $\theta_{\min}(m)$ denote the θ which satisfies

$$m\omega_n(\theta) = A_n(1) = 2\frac{\pi^{n/2}}{\Gamma(n/2)}.$$

Thus,

$$\theta_{\min}(m) = \omega_n^{-1}\left(\frac{2}{m}\frac{\pi^{n/2}}{\Gamma(n/2)}\right).$$

We see that $m\theta_{\min}(m)$-caps have as much total area as the surface area A_n of S_n; hence any placement of m such caps on S_n must involve overlapping.

Thus, if $x_1, \ldots, x_m$ are any unit vectors of R_n, there is an index pair $1 \leqslant i \neq j \leqslant m$ such that

$$(x_i|x_j) \geqslant \cos(\theta_{\min}(m)).$$

For this i and j,

$$||x_i - x_j|| \leqslant [2(1 - \cos(\theta_{\min}(m))]^{1/2} = N_n(m)$$

which completes the proof of (4).

Finally, we show that $N_n(m) \leqslant M_n(m)$ whenever m and n satisfy the conditions (2).

It has been shown (see [1, Lemma 1]) that, for $0 \leqslant \theta \leqslant \pi$,

$$\omega_n(\theta) \geqslant a_n(\theta) = \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}\sin^{n-1}(\theta/2).$$

(The inequality persists, of course, even when $\theta > \pi$.) Notice that $a_n(\cdot)$ is monotone increasing on $[0, \pi]$. Hence, if

$$0 = \min_{0\leqslant\theta\leqslant\pi} a_n(\theta) \leqslant \alpha \leqslant \max_{0\leqslant\theta\leqslant\pi} a_n(\theta) = \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}, \tag{8}$$

we have

$$0 \leqslant \omega_n^{-1}(\alpha) \leqslant a_n^{-1}(\alpha) \leqslant \pi.$$

The conditions (2) on m and n are enough to guarantee that (8) holds for the choice

$$\alpha = \frac{2}{m}\frac{\pi^{n/2}}{\Gamma(n/2)}$$

and so, under these conditions,

$$0 \leqslant \omega_n^{-1}\left(\frac{2}{m}\frac{\pi^{n/2}}{\Gamma(n/2)}\right) \leqslant a_n^{-1}\left(\frac{2}{m}\frac{\pi^{n/2}}{\Gamma(n/2)}\right)$$

$$= 2\sin^{-1}\left[\frac{2\sqrt{\pi}}{m}\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right]^{1/(n-1)} \leqslant \pi.$$

But now

$$N_n(m) = \left[2\left(1 - \cos\left[\omega_n^{-1}\left(\frac{2}{m}\frac{\pi^{n/2}}{\Gamma(n/2)}\right)\right]\right)\right]^{1/2}$$

$$\leqslant \left[2\left(1 - \cos\left[a_n^{-1}\left(\frac{2}{m}\frac{\pi^{n/2}}{\Gamma(n/2)}\right)\right]\right)\right]^{1/2}$$

$$= M_n(m).$$

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